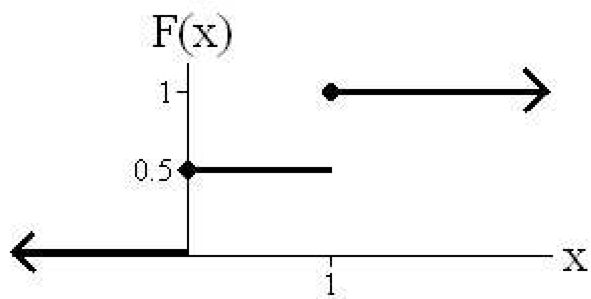
18.05 Lecture 9
February 23, 2005

Discrete Random Variable: - defined by probability function (p.f.)
$\{s_1, s_2, ...\}, f(s_i) = \mathbb{P}(X = s_i)$
Continuous: probability distribution function (p.d.f.) - also called density function.
$f(x) \geq 0, \int_{-\infty}^{\infty} f(x)dx, \mathbb{P}(X \in A) = \int_A f(x)dx$

Cumulative distribution function (c.d.f):
$F(x) = \mathbb{P}(X \leq x), x \in \mathbb{R}$
Properties:
1. $x_1 \leq x_2, \{X \leq x_1\} \subset \{X \leq x_2\}$
$\rightarrow \mathbb{P}(X \leq x_1) \leq \mathbb{P}(X \leq x_2)$ non-decreasing function.
2. $\lim_{x\rightarrow -\infty} F(x) = \mathbb{P}(X \leq -\infty) = 0, \lim_{x\rightarrow \infty} F(x) = \mathbb{P}(X \leq \infty) = 1.$
A random variable only takes real numbers, as $x \rightarrow -\infty$, set becomes empty.

Example: $\mathbb{P}(X = 0) = \frac{1}{2}, \mathbb{P}(X = 1) = \frac{1}{2}$



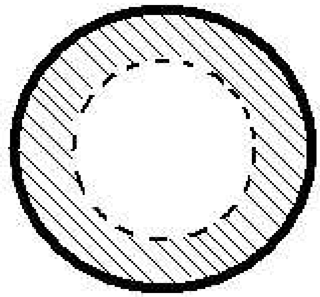
$\mathbb{P}(X \leq x < 0) = 0$
$\mathbb{P}(X \leq 0) = \mathbb{P}(X = 0) = \frac{1}{2}, \mathbb{P}(X \leq x) = \mathbb{P}(X = 0) = \frac{1}{2}, x \in [0, 1)$
$\mathbb{P}(X \leq x) = \mathbb{P}(X = 0 \text{ or } 1) = 1, x \in [1, \infty)$

3. "right continuous": $\lim_{y\rightarrow x^+} F(y) = F(x)$

$$F(y) = \mathbb{P}(X \leq y), \text{ event } \{X \leq y\}$$

$$\bigcap_{n=1}^{\infty} \{X \leq y_n\} = \{X \leq x\}, F(y_n) \rightarrow \mathbb{P}(X \leq x) = F(x)$$

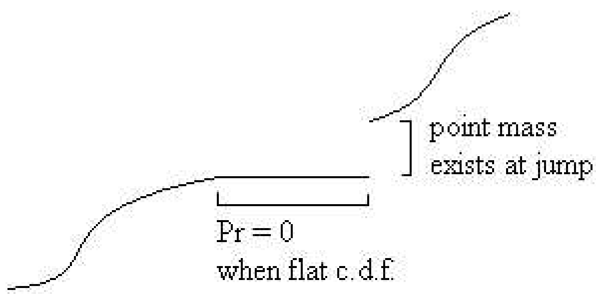
Probability of random variable occuring within interval:
$\mathbb{P}(x_1 < X < x_2) = \mathbb{P}(\{X \leq x_2\} \backslash \{X \leq x_1\})$
$= \mathbb{P}(X \leq x_2) - \mathbb{P}(X \leq x_1)$
$= F(x_2) - F(x_1)$



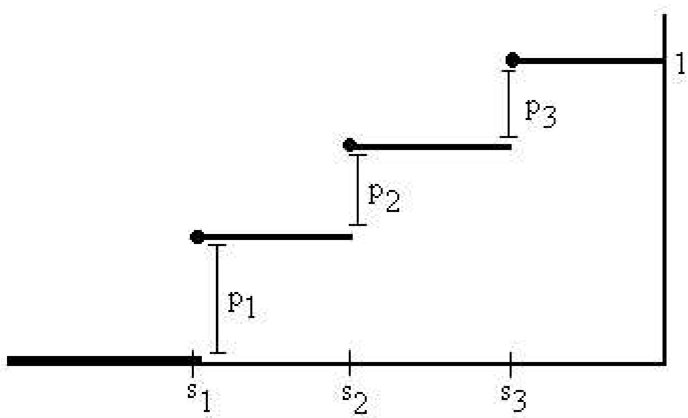
$\{X \leq x_2\} \supseteq \{X \leq x_1\}$

Probability of a point x, $\mathbb{P}(X = x)$
$= F(x) - F(x^-)$ where $F(x^-) = \lim_{x \to x^-} F(x), F(x^+) = \lim_{x \to x^+} F(x)$
If continuous, probability at a point is equal to 0, unless there is a jump, where the probability is the value of the jump.
$\mathbb{P}(x_1 \leq X \leq x_2) = F(x_2) - F(x_1^-)$
$\mathbb{P}(A) = \mathbb{P}(X \in A)$
X - random variable with distribution $\mathbb{P}$

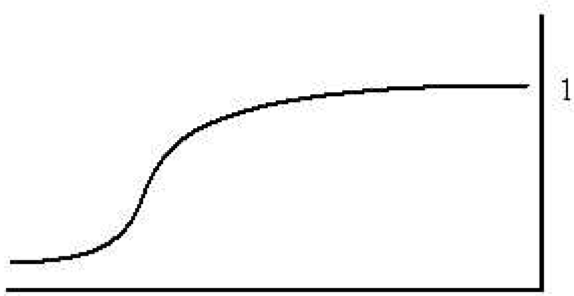
When observing a c.d.f:



Discrete: sum of probabilities at all the jumps = 1. Graph is horizontal in between the jumps, meaning that probability = 0 in those intervals.

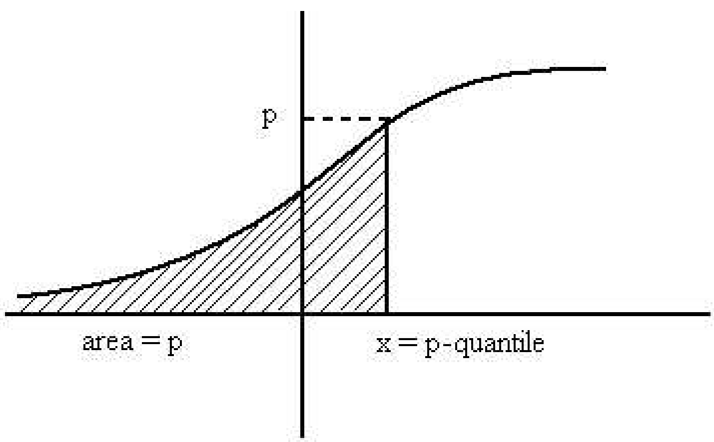


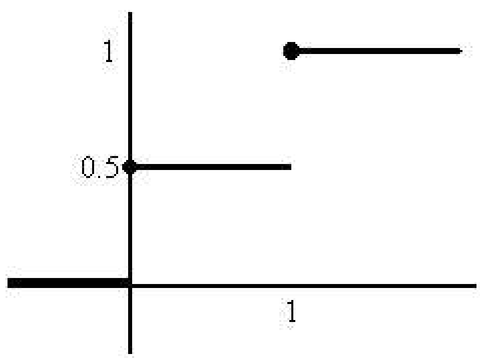
Continuous: $F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(x)dx$
eventually, the graph approaches 1.



If f continuous, $f(x) = F'(x)$

Quantile: $p \in [0, 1]$, p-quantile $= inf\{x : F(x) = \mathbb{P}(X \leq x) \geq p\}$
find the smallest point such that the probability up to the point is at least p.
The area underneath F(x) up to this point x is equal to p.
If the 0.25 quantile is at x = 0, $\mathbb{P}(X \leq 0) \geq 0.25$





Note that if disjoint, the 0.25 quantile is at x = 0,
but so is the 0.3, 0.4...all the way up to 0.5.

What if you have 2 random variables? multiple?
ex. take a person, measure weight and height. Separate behavior tells you nothing about the pairing, need to describe the joint distribution.
Consider a pair of random variables (X, Y)
Joint distribution of (X, Y): $\mathbb{P}((X, Y) \in A)$
Event, set A $\in \mathbb{R}^2$

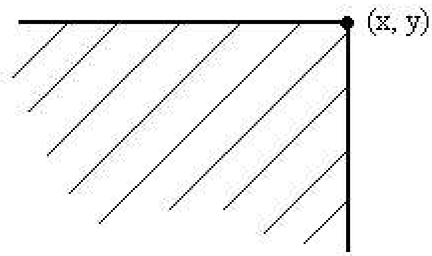
Discrete distribution: $\{(s_1^1, s_1^2), (s_2^1, s_2^2), ...\} \ni (X, Y)$
Joint p.f.: $f(s_i^1, s_i^2) = \mathbb{P}((X, Y) = (s_i^1, s_1^2))$
$= \mathbb{P}(X = s_i^1, Y = s_i^2)$
Often visualized as a table, assign probability for each point:

	0	-1	-2.5	5
1	0.1	0	0.2	0
1.5	0	0	0	0.1
3	0.2	0	0.4	0

Continuous:

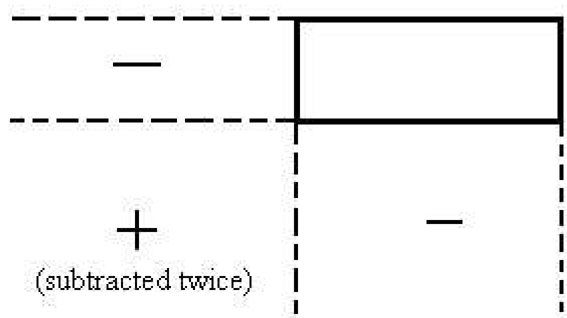
$$f(x, y) \geq 0, \int_{\mathbb{R}^2} f(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

Joint p.d.f. $f(x, y) : \mathbb{P}((X, Y) \in A) = \int_A f(x, y)dxdy$
Joint c.d.f. $F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$



If you want the c.d.f. only for x,
$F(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq x, Y \leq +\infty)$
$= F(x, \infty) = \lim_{y\to\infty} F(x, y)$
Same for y.

To find the probability within a rectangle on the (x, y) plane:



Continuous: $F(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x, y)dxdy$. Also, $\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$

** End of Lecture 9